Computers and Mathematics with Applications 73 (2017) 1363–1376

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Unidirectional flows of fractional Jeffreys' fluids: Thermodynamic constraints and subordination

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ARTICLE INFO

Article history:
Available online 9 January 2017

Keywords:
Viscoelastic fluid
Fractional Jeffreys' model
Stokes' first problem
Mittag-Leffler function
Completely monotone function

ABSTRACT

A class of initial–boundary value problems governing the velocity distribution of unidirectional flows of viscoelastic fluids is studied. The generalized fractional Jeffreys' constitutive model is used to describe the viscoelastic properties. Thermodynamic constraints on the parameters of the model are derived from the monotonicity of the corresponding relaxation function. Based on these constraints, a subordination principle for the considered class of problems is established. It gives an integral representation of the solution in terms of a probability density function and the solution of a related wave equation. Explicit representation of the probability density function is derived from the solution of the Stokes' first problem. Numerical verification of the obtained analytical results is provided.

1. Introduction

Increasing attention has been devoted to the prediction of behavior of viscoelastic non-Newtonian fluids in recent years, due to their various applications (molten plastics, oils and greases, suspensions, emulsions, etc.) The Jeffreys' fluid is a non-Newtonian rate-type fluid, which constitutive equation has the dimensionless form

$$\left(1 + a\frac{\partial}{\partial t}\right)\sigma = \left(1 + b\frac{\partial}{\partial t}\right)\dot{\varepsilon}, \tag{1}$$

where σ is the shear stress, $\dot{\varepsilon}$ is the rate of strain, a and b are the dimensionless relaxation and retardation times, respectively, with $a > b > 0$. This model has become very popular since it can describe many of the non-Newtonian characteristics of polymeric suspensions [1,2].

Fractional calculus has been extensively used in linear viscoelasticity [3–7]. Due to the non-locality of fractional derivatives, fractional order models provide a higher level of adequacy, preserving linearity, and give the possibility for relatively simple description of the complex behavior of non-Newtonian viscoelastic fluids. The fractional Jeffreys' model introduced in [8] (also referred to as fractional Oldroyd-B model) is derived from (1) by replacing the terms $a\frac{\partial}{\partial t}$ and $b\frac{\partial}{\partial t}$ by fractional differential operators aD_t^α and bD_t^β, respectively, with $0 < \alpha, \beta < 1$. In [8] a good fit with experimental data for this model is achieved.

Unidirectional flows of fractional Jeffreys' fluids in different geometries and under various conditions (e.g. porous media, magneto-hydrodynamic effects, slip conditions) are studied in [9–14], to mention only few of many recent publications. In these works solutions of the initial–boundary value problems (IBVPs) for the velocity distribution are obtained in the form

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http://dx.doi.org/10.1016/j.camwa.2016.12.009
0898-1221/

of series expansions. The considered IBVPs for unidirectional flows of fractional Jeffreys' fluids usually can be written in abstract form as

$$\left(1+aD_t^{\alpha}\right)u_t=(1+bD_t^{\beta})Au+\left(1+aD_t^{\alpha}\right)f(t),\quad t>0, \tag{2}$$

$$u(0)=u_t(0)=0, \tag{3}$$

where A is a linear operator defined in a suitably chosen Banach space X (A is usually a one- or two-dimensional realization of the Laplace operator, or a more general elliptic operator) and f is an X-valued function.

Regarding the orders α and β of the fractional derivatives different assumptions are considered in the literature: e.g. in [8] the restriction $\alpha \geq \beta$ is imposed, in [12,13] it is assumed that $\alpha \leq \beta$, while in other works the whole range $0<\alpha,\beta<1$ is considered. On the other hand, applying the technique developed by Bagley and Torvik in [4], it is proven in [15] that the fractional Jeffreys' model is consistent with the second law of thermodynamics if and only if

$$\alpha=\beta \quad \text{and} \quad a\geq b. \tag{4}$$

However, to the best of our knowledge, the impact of thermodynamic restrictions (4) on the solution of the corresponding problem (2)–(3) has not been analyzed.

Stokes' first problem is concerned with a specific case of a unidirectional flow set into motion by a sudden movement of a flat plate. One of the first studies on Stokes' first problem for classical Jeffreys' fluids is [16]. Since then many works have been devoted to this problem and its fractional order generalizations. As pointed out in [17,18], in a number of recent works the obtained exact solutions of Stokes' first problem for classical Jeffreys' fluids contain mistakes, which propagate as well to their fractional order generalizations.

Motivated by the aforementioned developments, in this work we first revisit the fractional order Jeffreys' constitutive model and the Stokes' first problem for thermodynamically compatible fractional Jeffreys' fluids. Thermodynamic constraints (4) are derived from the monotonicity properties of the corresponding relaxation function and exact solutions for the Stokes' first problems in half-space and on a strip are obtained. Next, the general problem (2)–(3) is studied, where the operator A is a generator of a cosine family (for definition see e.g. [19], Section 3.14). Our main result is the following subordination representation for the solution operator $S(t)$ of this problem in the case when thermodynamic constraints (4) are satisfied:

$$S(t)=\int_0^{\infty}\varphi(t,\tau)T(\tau)\,\mathrm{d}\tau,\quad t>0, \tag{5}$$

where $\varphi(t,\tau)$ is a probability density function (p.d.f.) in τ, and $T(t)$ is the cosine family generated by the operator A. By the variation of parameters formula the solution of (2)–(3) is then given by

$$u(t)=\int_0^{t}S(t-\sigma)f(\sigma)\,\mathrm{d}\sigma. \tag{6}$$

The p.d.f. $\varphi(t,\tau)$ can be expressed via the solution of the Stokes' first problem in half-space. This provides an explicit integral representation for $\varphi(t,\tau)$.

Subordination principle in a general setting is introduced in [20], Chapter 4. Various applications of this principle have been found so far: in inverse problems [21], asymptotic analysis of fractional diffusion-wave equations [22], stochastic solutions [23], abstract Volterra integro-differential equations [24], operator theory [25], etc.

This paper is organized as follows. In Section 2 the relaxation function of the fractional Jeffreys' model is studied and the thermodynamic constraints (4) are derived from its monotonicity. In Section 3 explicit solutions for Stokes' first problems in half-space and on a strip are derived and their properties are discussed. Section 4 is devoted to the derivation of the subordination formula and its applications. Numerical results based on the obtained analytical representations are given in Section 5 and independent numerical checks are performed. Some facts concerning completely monotone functions and related classes of functions are summarized in an Appendix.

2. Thermodynamic restrictions and relaxation function

Consider a unidirectional viscoelastic flow and suppose it is quiescent for all times prior to some starting time that we assume as $t=0$. Since we work only with causal functions ($f(t)=0$ for $t<0$) if there is no danger of confusion for the sake of brevity we still denote by $f(t)$ the function $H(t)f(t)$, where $H(t)$ is the Heaviside unit step function.

According to the generalized fractional Jeffreys' model the relationship between stress $\sigma(t)$ and strain $\varepsilon(t)$ is given by the following linear constitutive equation [8,15]

$$\left(1+aD_t^{\alpha}\right)\sigma(t)=(1+bD_t^{\beta})\dot{\varepsilon}(t) \tag{7}$$

where $a,b>0$, $0<\alpha,\beta\leq 1$, the over-dot denotes the first time derivative, and D_t^{α}, D_t^{β} are Riemann–Liouville fractional time derivatives:

$$D_t^{\gamma}f(t)=\frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{t}\frac{f(\tau)}{(t-\tau)^{\gamma}}\,\mathrm{d}\tau,\quad \gamma\in(0,1),$$

where $\Gamma(\cdot)$ is the Gamma function, $D_t^1=\mathrm{d}/\mathrm{d}t$.

The relaxation function $G(t)$ in a one-dimensional linear viscoelastic model is defined by the identity [5,6]

$$\sigma(t) = \int_0^t G(t-\tau)\dot{\varepsilon}(\tau)\,d\tau. \tag{8}$$

The function $G(t)$ should be non-negative and non-increasing for $t > 0$, which is related to the physical phenomenon of stress relaxation.

The constitutive equation is conveniently treated by the use of Laplace transform

$$\mathcal{L}\{f(t)\}(s) = \widehat{f}(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Applying Laplace transform in Eqs. (7) and (8) and using the property

$$\mathcal{L}\{D_t^\gamma f\}(s) = s^\gamma \widehat{f}(s), \quad 0 < \gamma < 1, \tag{9}$$

for functions satisfying $\lim_{t\to 0^+} f(t) < \infty$ (see e.g. [5]) one obtains the following identity for the relaxation function of the fractional Jeffreys' model

$$\widehat{G}(s) = \frac{1+bs^\beta}{1+as^\alpha}. \tag{10}$$

Taking inverse Laplace transform an explicit expression for $G(t)$ is derived:

$$G(t) = \frac{1}{a}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{1}{a}t^\alpha\right) + \frac{b}{a}t^{\alpha-\beta-1}E_{\alpha,\alpha-\beta}\left(-\frac{1}{a}t^\alpha\right), \tag{11}$$

where $E_{\alpha,\beta}(.)$ denotes the two-parameter Mittag-Leffler function [26,27]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{12}$$

Here we have used the identity

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)\right\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha+\lambda}. \tag{13}$$

According to the procedure presented in [4] model (7) is consistent with the second law of thermodynamics if and only if the real and the imaginary parts of $\mathrm{i}\omega\widehat{G}(\mathrm{i}\omega)$ are non-negative for $\omega > 0$:

$$\Re\left\{\mathrm{i}\omega\frac{1+b(\mathrm{i}\omega)^\beta}{1+a(\mathrm{i}\omega)^\alpha}\right\} \ge 0, \qquad \Im\left\{\mathrm{i}\omega\frac{1+b(\mathrm{i}\omega)^\beta}{1+a(\mathrm{i}\omega)^\alpha}\right\} \ge 0, \quad \forall\omega > 0. \tag{14}$$

In the next theorem we formulate other conditions, which are necessary and sufficient for thermodynamic compatibility of model (7). In particular, we derive thermodynamic restrictions (4) from the monotonicity properties of the relaxation function.

Definitions and properties of completely monotone functions and related classes of functions are given in the Appendix.

Theorem 2.1. *Assume $\alpha, \beta \in (0, 1)$, $a, b > 0$. The following assertions are equivalent:*

(a) *Inequalities (14) are satisfied;*
(b) *$\alpha = \beta$ and $a \ge b$;*
(c) *$G(t)$ is non-negative for $t > 0$;*
(d) *$G(t)$ is non-increasing for $t > 0$;*
(e) *$G(t)$ is completely monotone function for $t > 0$.*

If any of the above conditions is satisfied then $G(t)$ admits the representation

$$G(t) = \mu\delta(t) + (1-\mu)\frac{1}{a}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{1}{a}t^\alpha\right), \quad \mu = \frac{b}{a}, \tag{15}$$

where $\delta(t)$ is the Dirac delta function.

Proof. The equivalence of (a) and (b) is established in [15]. We will prove that (b) is equivalent to any of the conditions (c)–(e).

First we prove that any of conditions (c) and (d) implies $\alpha = \beta$. Indeed, if we assume that $\alpha < \beta$, then taking the first terms of the expansions of the Mittag-Leffler functions in (11) we obtain

$$G(t) \approx \frac{b}{a} \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}, \quad t \to 0. \tag{16}$$

Since in this case $-1 < \alpha - \beta < 0$ and thus $\Gamma(\alpha-\beta) < 0$ it follows from (16) that any of conditions (c) and (d) is violated. On the other hand, if we suppose that $\alpha > \beta$ then the asymptotic expansion of the Mittag-Leffler function [26,27]

$$E_{\alpha,\beta}(-t) = -\sum_{k=1}^{N-1} \frac{(-t)^{-k}}{\Gamma(\beta-\alpha k)} + O(t^{-N}), \quad t \to +\infty, \tag{17}$$

implies

$$G(t) \approx b\frac{t^{-\beta-1}}{\Gamma(-\beta)}, \quad t \to +\infty,$$

which indicates violation of conditions (c) and (d) for large t. Therefore $\alpha = \beta$. To prove that also $a \geq b$ we deduce representation (15) first. To this end we take $\alpha = \beta$ in (10) and obtain

$$\widehat{G}(s) = \frac{1+bs^\alpha}{1+as^\alpha} = \frac{b}{a} + \frac{1}{a}\left(1-\frac{b}{a}\right)\frac{1}{s^\alpha + 1/a}. \tag{18}$$

Applying inverse Laplace transform to (18) and using the identities (13) and $\mathcal{L}\{\delta(t)\} = 1$ we deduce representation (15). Further, since $t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha/a)$ is completely monotone for $t > 0$ (see e.g. [27]), representation (15) shows that any of (c) and (d) implies $a \geq b$. In this way we finished the nontrivial part of the proof: that any of the conditions (c) and (d) implies (b). In addition, representation (15) shows that (b) implies (e). To finish the proof, we note that (e) implies both (c) and (d) by definition. □

In what follows we consider only thermodynamically compatible models, i.e. we assume that the properties listed in Theorem 2.1 are satisfied.

Representation (15) and the asymptotic expansion (17) give

$$G(t) = O(t^{-\alpha-1}), \quad t \to +\infty,$$

i.e. $\lim_{t\to+\infty} G(t) = 0$ and $G(t)$ is integrable on $(0, \infty)$. In fact

$$\int_0^\infty G(t)\,\mathrm{d}t = 1,$$

which follows from (15) by applying the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\alpha,1}\left(-\frac{1}{a}t^\alpha\right) = -\frac{1}{a}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{1}{a}t^\alpha\right). \tag{19}$$

This behavior of the relaxation function confirms that the fractional Jeffreys' constitutive equation indeed models fluid-like behavior (for a discussion on the general conditions see [5], Section 2).

3. Stokes' first problem

Consider a plane Couette flow of an incompressible viscoelastic fluid with the thermodynamically compatible fractional Jeffreys' constitutive equation

$$\left(1 + aD_t^\alpha\right)\sigma(t) = (1 + bD_t^\alpha)\dot{\varepsilon}(t), \quad a \geq b > 0. \tag{20}$$

Assume the fluid fills a half-space $x > 0$ and is set into motion by a sudden movement of the bounding plane $x = 0$. Denote by $u(x,t)$ the induced velocity field. Noting that $\dot{\varepsilon} = \partial u/\partial x$ and eliminating σ between Eq. (20) and Cauchy's first law $\partial u/\partial t = \partial\sigma/\partial x$ (see e.g. [16]) we obtain the following IBVP

$$\left(1 + aD_t^\alpha\right)u_t(x,t) = (1 + bD_t^\alpha)u_{xx}(x,t), \quad x, t > 0, \tag{21}$$

$$u(x,0) = u_t(x,0) = 0, \quad x > 0, \tag{22}$$

$$u(0,t) = H(t), \quad u \to 0 \text{ as } x \to \infty, \ t > 0, \tag{23}$$

where $H(t)$ is the Heaviside unit step function.

In one variation of the above problem the fluid is confined between parallel plates, which we take at $x = 0$ and $x = 1$. Then the boundary conditions (23) are replaced by

$$u(0,t) = H(t), \qquad u(1,t) = 0, \quad t > 0. \tag{24}$$

In this paper, problem (21)–(22)–(23) is referred to as Stokes' first problem in a half-space and problem (21)–(22)–(24) as Stokes' first problem on a strip.

3.1. Stokes' first problem in a half-space

By applying Laplace transform with respect to the temporal variable in (21) and (23) and using (22) we obtain for the Laplace transform of $u(x,t)$ w.r.t. t

$$\widehat{u}(x,s) = \frac{1}{s}\exp\left(-x\sqrt{g(s)}\right), \tag{25}$$

where

$$g(s) = \frac{s(1+as^\alpha)}{1+bs^\alpha}. \tag{26}$$

For derivation details see e.g. [9].

To find explicit integral expression for the solution $u(x,t)$ we apply Bromwich integral inversion formula to (25):

$$u(x,t) = \frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \frac{1}{s}\exp\left(st - x\sqrt{g(s)}\right)\,\mathrm{d}s, \quad \gamma > 0 \tag{27}$$

and use the technique of [28]. By the Cauchy's theorem, the integration on the Bromwich path can be replaced by integration on the contour $D \cup D_0$, where

$$D = \{s = \mathrm{i}r,\ r \in (-\infty,-\varepsilon) \cup (\varepsilon,\infty)\}, \qquad D_0 = \{s = \varepsilon e^{\mathrm{i}\theta}, \theta \in [-\pi/2, \pi/2]\}.$$

This is possible since the integrals on the contours $\{s = \sigma \pm \mathrm{i}R, \sigma \in (0,\gamma)\}$ vanish for $R \to \infty$ due to the following asymptotic expression

$$\Re\sqrt{g(s)} \approx \sqrt{\frac{a}{b}|s|}\cos\frac{\arg s}{2} \approx \left(\frac{a}{b}(\sigma^2+R^2)^{1/2}\right)^{1/2}\cos(\pm\pi/4), \quad R \to \infty.$$

Further, since

$$\lim_{s\to 0} s\left(\frac{1}{s}\exp\left(st - x\sqrt{g(s)}\right)\right) = 1,$$

applying Jordan's lemma it follows that the integral on the semi-circular contour D_0 equals $1/2$. Integration on the contour D yields after letting $\varepsilon \to 0$ and $R \to \infty$:

$$\frac{1}{2\pi\mathrm{i}}\int_D \frac{1}{s}\exp\left(st - x\sqrt{g(s)}\right)\,\mathrm{d}s = \frac{1}{\pi}\int_0^\infty \frac{1}{r}\Im\exp\left(\mathrm{i}rt - x\sqrt{g(\mathrm{i}r)}\right)\,\mathrm{d}r.$$

Applying the formula for real and imaginary parts of the square root of a complex number we obtain after some standard manipulations the following result.

Theorem 3.1. *The solution of the Stokes' first problem in a half-space (21)–(22)–(23) admits the integral representation:*

$$u(x,t) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \exp(-xK^-(r))\sin(rt - xK^+(r))\,\frac{\mathrm{d}r}{r}, \quad x,t > 0, \tag{28}$$

where

$$K^\pm(r) = \left(\frac{r}{2}\right)^{1/2}\left(\left(A^2(r)+B^2(r)\right)^{1/2} \pm A(r)\right)^{1/2} \tag{29}$$

with

$$A(r) = \frac{(a-b)r^\alpha\sin(\alpha\pi/2)}{1+2br^\alpha\cos(\alpha\pi/2)+b^2r^{2\alpha}},$$

$$B(r) = \frac{1+(a+b)r^\alpha\cos(\alpha\pi/2)+abr^{2\alpha}}{1+2br^\alpha\cos(\alpha\pi/2)+b^2r^{2\alpha}}.$$

We prove next that the solution to the Stokes' first problem in a half-space has a physically acceptable behavior: it is non-negative and non-increasing w.r.t. x. This follows from an important property of the function $g(s)$ defined in (26), which is established in the next proposition. The definition of Bernstein functions as well as other definitions, notations and properties used in the proof of the proposition are given in the Appendix.

Proposition 3.2. *Let $a \geq b > 0$ and $0 < \alpha \leq 1$. If $g(s)$ is defined by (26) then $\sqrt{g(s)}$ is a Bernstein function.*

Proof. Consider the function $f(s) = \sqrt{g(s)}/s$. Then

$$f^2(s) = \frac{1 + as^\alpha}{s(1 + bs^\alpha)}$$

is a Stieltjes function since, according to (13), it is Laplace transform of the completely monotone function $1 + (a/b - 1)E_{\alpha,1}(t^\alpha/b)$. On the other hand, $s^{1/2}$ is a complete Bernstein function due to the fact that $s^{-1/2} = \mathcal{L}\{t^{-1/2}/\Gamma(1/2)\}$ is a Stieltjes function. This together with property (D) from the Appendix implies that $f(s)$ is a Stieltjes function and thus $\sqrt{g(s)} = sf(s)$ is a complete Bernstein function, in particular it is a Bernstein function. □

Let us note that $g(s)$ itself is not a Bernstein function for $a > b$. Indeed, for small s we have $g(s) \approx s + (a - b)s^{\alpha+1}$ and thus $g''(s) \approx (a - b)(\alpha + 1)\alpha s^{\alpha-1} > 0$.

Proposition 3.2 implies the following result.

Theorem 3.3. *The solution of the Stokes' first problem in a half-space (21)–(22)–(23) satisfies*

$$u(x, t) \geq 0, \quad \frac{\partial}{\partial x} u(x, t) \leq 0, \ x, t > 0. \tag{30}$$

Proof. By Proposition 3.2 and property (C) in the Appendix the function $\exp(-x\sqrt{g(s)})$ is completely monotone as a composition of the completely monotone in λ function $e^{-x\lambda}$ (for fixed $x > 0$) and the Bernstein function $\sqrt{g(s)}$. In addition, $1/s$ and $\sqrt{g(s)}/s$ are completely monotone (for the second function this follows from property (B) in the Appendix). Therefore, the function $\widehat{u}(x, s)$ given in (25) and

$$-\frac{\partial}{\partial x}\widehat{u}(x, s) = \frac{\sqrt{g(s)}}{s} \exp\left(-x\sqrt{g(s)}\right) \tag{31}$$

are completely monotone functions for $s > 0$ as products of two completely monotone functions. Applying the Bernstein's theorem with respect to the temporal variable we infer inequalities (30). □

3.2. Stokes' first problem on a strip

For this problem we apply standard eigenfunction expansion technique as in [10,17,29].

Theorem 3.4. *The solution of the Stokes' first problem on a strip (21)–(22)–(24) admits the representation*

$$u(x, t) = 1 - x - \frac{2}{\pi} \sum_{n=1}^{\infty} S_n(t) \frac{\sin(n\pi x)}{n}, \tag{32}$$

where

$$S_n(t) = \exp(-\mu\lambda_n t) + \sum_{k=1}^{\infty} \sum_{m=1}^{k} (-\lambda_n)^k \binom{k}{m} \mu^{k-m} \left(\frac{1-\mu}{a}\right)^m t^{\alpha m+k} E^m_{\alpha,\alpha m+k+1}\left(-\frac{1}{a}t^\alpha\right) \tag{33}$$

with $\mu = b/a$ and $E^m_{\alpha,\beta}(.)$—the three-parameter Mittag-Leffler function

$$E^m_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} \frac{(m)_j}{j!}, \tag{34}$$

$(m)_j = m(m + 1) \cdots (m + j - 1)$, $(m)_0 = 1$.

Proof. The solution $u(x, t)$ of problem (21)–(22)–(24) is represented as a sum of the steady-state solution $1 - x$ and the solution of the following problem with homogeneous boundary conditions (see e.g. [29])

$$\begin{aligned} &\left(1 + aD_t^\alpha\right) u_t(x, t) = (1 + bD_t^\alpha) u_{xx}(x, t) - \left(1 + aD_t^\alpha\right) \delta(t)(1 - x); \\ &u(0, t) = u(1, t) = 0, \quad t > 0; \qquad u(x, 0) = u_t(x, 0) = 0, \quad 0 < x < 1. \end{aligned}$$

To solve this problem we apply eigenfunction decomposition and Laplace transform technique for the obtained equations in the eigenspaces. Since the procedure we follow is a straightforward generalization of the one described and discussed in [17] in the case $\alpha = 1$, the details are omitted here. In this way we deduce representation (32) where the functions $S_n(t)$ are defined by their Laplace transforms as follows

$$\widehat{S_n}(s) = \frac{1}{s}\left(1 + \frac{\lambda_n(1+bs^\alpha)}{s(1+as^\alpha)}\right)^{-1}, \quad \lambda_n = n^2\pi^2. \tag{35}$$

Using the same rearrangement as in (18) we rewrite expression (35) for large $|s|$ in a series form

$$\widehat{S_n}(s) = \sum_{k=0}^{\infty}\sum_{m=0}^{k}(-\lambda_n)^k \binom{k}{m} \mu^{k-m}\left(\frac{1-\mu}{a}\right)^m \frac{s^{-k-1}}{(s^\alpha+1/a)^m}, \tag{36}$$

where $\mu = b/a$. Taking the inverse Laplace transform in (36) and using the identity (see e.g. [27])

$$\mathcal{L}\{t^{\beta-1}E^m_{\alpha,\beta}(-\nu t^\alpha)\}(s) = \frac{s^{\alpha m-\beta}}{(s^\alpha+\nu)^m}$$

the representation (33) is deduced. □

Let us note that in the limiting case $a = b$ ($\mu = 1$) corresponding to a Newtonian fluid Eq. (33) reduces to $S_n(t) = \exp(-\lambda_n t)$.

We finish this subsection with some comments on the behavior of the eigenmodes $S_n(t)$. First, (33) implies that $S_n(0) = 1$, more precisely

$$S_n(t) \approx 1 - \mu\lambda_n t, \quad t \to 0.$$

Further, it follows from (35)

$$\widehat{S_n}(s) \approx \frac{1+as^\alpha}{\lambda_n(1+bs^\alpha)} = \frac{1}{\lambda_n}\left(\frac{a}{b} + \frac{b-a}{b}\frac{1}{1+bs^\alpha}\right), \quad |s| \to 0,$$

and taking the inverse Laplace transform yields

$$S_n(t) \approx \frac{b-a}{\lambda_n b^2} t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{1}{b}t^\alpha\right), \quad t \to +\infty,$$

which by the use of (17) gives the asymptotic behavior of the eigenmodes for large t

$$S_n(t) \approx \frac{a-b}{\lambda_n \Gamma(-\alpha)t^{\alpha+1}}, \quad t \to +\infty. \tag{37}$$

Therefore, the functions $S_n(t)$ admit the following behavior: starting from 1 at $t = 0$, after some oscillations, they become permanently negative and vanish.

Asymptotic expansion (37) is helpful for the numerical computation of $S_n(t)$ for large t, since Eq. (33) is not appropriate for this purpose.

In the next section, another representation for the eigenmodes $S_n(t)$ is derived as a particular case of the formulated there subordination identity.

4. Subordination

Consider a generalization of Eq. (21), in which the second-order space derivative is replaced by a general linear closed operator A densely defined on a suitable Banach space X. We are concerned with the Cauchy problem for this equation

$$\left(1 + aD_t^\alpha\right)u_t = (1 + bD_t^\alpha)Au, \quad t > 0, \tag{38}$$

$$u(0) = v \in X, \qquad u_t(0) = 0. \tag{39}$$

Consider also the second-order Cauchy problem

$$w_{tt} = Aw, \quad t > 0, \tag{40}$$

$$w(0) = v \in X, \qquad w_t(0) = 0. \tag{41}$$

We shall prove that the solution of (38)–(39) is related to the solution of (40)–(41) via the following subordination identity

$$u(t) = \int_0^\infty \varphi(t,\tau)w(\tau)\,d\tau, \quad t > 0, \tag{42}$$

where $\varphi(t,\tau)$ is a p.d.f. in τ, that is

$$\varphi(t,\tau)\geq 0,\quad \int_0^\infty \varphi(t,\tau)\,\mathrm{d}\tau=1. \tag{43}$$

A relationship of the form (42) is a manifestation of the so-called subordination principle. This principle is studied in [20], Chapter 4, in the case of abstract Volterra integral equations. Therefore, it will be convenient to rewrite problem (38)–(39) as a Volterra integral equation. Applying Laplace transform to (38) we obtain by the use of (9)

$$\widehat{u_t}(s)=\frac{1+bs^\beta}{1+as^\alpha}A\widehat{u}(s),$$

which together with identity (10) implies the following integro-differential equation

$$u_t(t)=\int_0^t G(t-\tau)Au(\tau)\,\mathrm{d}\tau,\quad u(0)=v, \tag{44}$$

where $G(t)$ is the relaxation function (15). Integrating (44) once again, we deduce the Volterra integral equation

$$u(t)=v+\int_0^t k(t-\tau)Au(\tau)\,\mathrm{d}\tau, \tag{45}$$

with kernel $k(t)=\int_0^t G(\tau)\,\mathrm{d}\tau$. Using identities (15) and (19) the following explicit representation for the kernel is derived

$$k(t)=1-\left(1-\frac{b}{a}\right)E_{\alpha,1}\left(-\frac{1}{a}t^\alpha\right). \tag{46}$$

In this section we investigate problem (38)–(39) in the form of Volterra integral equation (45) with kernel (46).

To make subordination identity (42) more convenient for applications we formulate it in terms of solution operators (for the precise definition see [20, Definition 1.3]). Denote by $S(t)$ the solution operator of the Volterra integral equation (45). Applying Laplace transform in (45) it follows by the use of (46) and (13) that

$$\int_0^\infty e^{-st}S(t)\,\mathrm{d}t=\frac{g(s)}{s}(g(s)-A)^{-1}, \tag{47}$$

where

$$g(s)=\frac{1}{\widehat{k}(s)}=\frac{s(1+as^\alpha)}{1+bs^\alpha}.$$

Note that this is the same function as the one defined in (26).

Denote by $T(t)$ the solution operator of problem (40)–(41). It is also known as cosine family generated by the operator A and its Laplace transform satisfies the identity

$$\int_0^\infty e^{-st}T(t)\,\mathrm{d}t=s(s^2-A)^{-1}. \tag{48}$$

For more details on cosine families we refer to [19], Section 3.14.

So, our aim is to prove that there exists a function $\varphi(t,\tau)$ satisfying (43) such that

$$S(t)=\int_0^\infty \varphi(t,\tau)T(\tau)\,\mathrm{d}\tau,\quad t>0, \tag{49}$$

and to find an explicit representation for this function.

Take $\varphi(t,\tau)$ such that its Laplace transform w.r.t. t

$$\widehat{\varphi}(s,\tau)=\int_0^\infty e^{-st}\varphi(t,\tau)\,\mathrm{d}t,\quad s,\tau>0,$$

satisfies the identity

$$\widehat{\varphi}(s,\tau)=\frac{\sqrt{g(s)}}{s}\exp\left(-\tau\sqrt{g(s)}\right),\quad s,\tau>0. \tag{50}$$

Define an operator-valued function $S(t)$ by (49). Application of the Laplace transform in (49) gives by the use of (50) and (48)

$$\begin{aligned}\int_0^\infty e^{-st}S(t)\,\mathrm{d}t &= \int_0^\infty \widehat{\varphi}(s,\tau)T(\tau)\,\mathrm{d}\tau\\ &= \frac{\sqrt{g(s)}}{s}\int_0^\infty \exp\left(-\tau\sqrt{g(s)}\right)T(\tau)\,\mathrm{d}\tau\\ &= \frac{g(s)}{s}(g(s)-A)^{-1}.\end{aligned} \tag{51}$$

Comparing this result to (47), it follows by the uniqueness of the Laplace transform that $S(t)$ is exactly the solution operator of (45).

To prove that $\varphi(t,\tau)$ defined by (50) is a p.d.f. we note first that by Bromwich inversion formula

$$\varphi(t,\tau)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\frac{\sqrt{g(s)}}{s}\exp\left(st-\tau\sqrt{g(s)}\right)\,\mathrm{d}s,\quad \gamma,t,\tau>0, \tag{52}$$

and therefore

$$\begin{aligned}\int_0^\infty\varphi(t,\tau)\mathrm{d}\tau &= \frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}e^{st}\frac{\sqrt{g(s)}}{s}\int_0^\infty\exp\left(-\tau\sqrt{g(s)}\right)\,\mathrm{d}\tau\mathrm{d}s\\ &=\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\frac{e^{st}}{s}\mathrm{d}s=1.\end{aligned}$$

Further, it is proven in Theorem 3.3 that the inverse Laplace transform of function (31) is non-negative. Comparing (50) to (31) it follows that $\varphi(t,\tau)\geq 0$ and that this function is related to the solution $u(x,t)$ of the Stokes' first problem in half-space via the identity

$$\varphi(t,\tau)=-\frac{\partial}{\partial x}u(x,t)\bigg|_{x=\tau},\quad t,\tau>0, \tag{53}$$

which together with the expression (28) leads to an explicit representation for $\varphi(t,\tau)$.

The fact that $\varphi(t,\tau)$ is a p.d.f. has a straightforward implication: if $T(t)$ is bounded, $\|T(t)\|\leq M$, $t\geq 0$, then the same holds for $S(t)$:

$$\|S(t)\|\leq\int_0^\infty\varphi(t,\tau)\|T(\tau)\|\,\mathrm{d}\tau\leq M\int_0^\infty\varphi(t,\tau)\,\mathrm{d}\tau=M,\quad t\geq 0.$$

Summarizing we have the following result.

Theorem 4.1. *Let $a\geq b>0$ and $0<\alpha\leq 1$. Assume A is a generator of a bounded cosine family $T(t)$ in X. Then problem (45) admits a bounded solution operator $S(t)$. It is related to $T(t)$ by the identity (49), which holds in the strong sense.*

The function $\varphi(t,\tau)$ is a p.d.f. in τ (i.e. conditions (43) hold) and admits the integral representation

$$\varphi(t,\tau)=\frac{1}{\pi}\int_0^\infty\exp\left(-\tau K^-(r)\right)\left(K^-(r)\sin\left(rt-\tau K^+(r)\right)+K^+(r)\cos\left(rt-\tau K^+(r)\right)\right)\frac{\mathrm{d}r}{r},\quad t,\tau>0, \tag{54}$$

where $K^{\pm}(r)$ are defined in (29).

Proof. The strict proof of the existence of solution operator $S(t)$ follows from Theorem 4.3 (ii) in [20]. According to this theorem it suffices to prove that $k(t)$ is a creep function (i.e. it is non-negative, non-decreasing and concave) and that the function $k_1(t)=\dot{k}(t)-\lim_{t\to\infty}k(t)/t$ is log-convex.

It is clear from (46) that $k(t)$ is a Bernstein function, therefore it is a creep function. Moreover,

$$k_1(t)=\frac{1}{a}\left(1-\frac{b}{a}\right)t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{1}{a}t^\alpha\right)$$

is a completely monotone function, in particular $k_1(t)$ is log-convex (see [30], Theorem 2.8).

Integral representation (54) follows from (53) and (28). □

Let us note that the function $\varphi(t,\tau)$ appears also in [7], Ch. 7, where another representation is given, see eq. [7.61].

The properties of $k(t)$ established in the proof of Theorem 4.1 ($k(t)$—creep function with $k_1(t)$ log-convex) imply that $(\widehat{k}(s))^{-1/2}$ is a Bernstein function (see Lemma 4.2 in [20]). This gives an alternative proof of Proposition 3.2.

From the arguments preceding Theorem 4.1 we infer that a general Volterra integral equation (45) is subordinate to problem (40)–(41) provided $(\widehat{k}(s))^{-1/2}$ is a Bernstein function (see also the discussion in [20], p. 103).

In the limiting case $a=b$ problem (38)–(39) reduces to the classical first-order Cauchy problem and (49) is known as the abstract Weierstrass formula. In this case

$$\varphi(t,\tau)=\frac{1}{\sqrt{\pi t}}\exp\left(-\tau^2/(4t)\right).$$

Next we give two applications of Theorem 4.1.

First, we use subordination identity (49) to find integral representations of the eigenmodes $S_n(t)$. Consider the simplest scalar case $X=\mathbb{R}$ and operator A being defined as multiplication by a scalar $-\lambda$, $\lambda>0$, take $v=1$. The solution of (38)–(39)

for $\lambda = \lambda_n$ is exactly the eigenmode $S_n(t)$. On the other hand, the solution of (40)–(41) for $\lambda = \lambda_n$ is $\cos(n\pi t)$. Therefore, in this case formula (49) implies

$$S_n(t) = \int_0^\infty \varphi(t,\tau)\cos(n\pi\tau)\,\mathrm{d}\tau \tag{55}$$

with $\varphi(t,\tau)$ defined in (54). This integral representation of the eigenmodes $S_n(t)$ is appropriate for numerical computation.

Second, consider a problem governing the velocity distribution of a plane Poiseuille flow: e.g. flow between two parallel plates set in motion due to sudden application of a constant pressure gradient (P). The corresponding IBVP is (see e.g. [10] for derivation details):

$$\begin{aligned} &\left(1 + aD_t^\alpha\right) u_t = (1 + bD_t^\alpha) u_{xx} + \left(1 + aD_t^\alpha\right) P, \quad 0 < x < 1, t > 0, \\ &u(0,t) = u(1,t) = 0, \quad t > 0; \qquad u(x,0) = u_t(x,0) = 0, \quad 0 < x < 1. \end{aligned} \tag{56}$$

Let us set now $X = L^2([0,1])$ and define A by $(Au)(x) = u''(x),\ x \in [0,1]$ with domain $D(A) = \{u \in X : u', u'' \in X, u(0) = u(1) = 0\}$. The corresponding cosine family $T(t)$ is defined by the solution of the following problem for the wave equation

$$\begin{aligned} &w_{tt} = w_{xx}, \quad 0 < x < 1, t > 0, \\ &w(0,t) = w(1,t) = 0, \quad t > 0; \qquad w(x,0) = v(x), \qquad w_t(x,0) = 0, \quad 0 < x < 1. \end{aligned} \tag{57}$$

Therefore, if $v(x)$ has the eigenexpansion $v(x) = \sum_{n=1}^\infty v_n \sin(n\pi x)$ then

$$(T(t)v)(x) = \sum_{n=1}^\infty v_n \sin(n\pi x)\cos(n\pi t). \tag{58}$$

From the subordination identity (49) and the variation of parameters formula (6) we have

$$u(x,t) = \int_0^t S(\tau) P \mathrm{d}\tau = \int_0^t \int_0^\infty \varphi(\tau,\sigma) T(\sigma) P\,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{59}$$

Here $T(t)P$ is the solution of problem (57) with $v = P$. Applying (58) it follows

$$T(t)P = \frac{2P}{\pi}\sum_{n=1}^\infty \frac{1-(-1)^n}{n}\sin(n\pi x)\cos(n\pi t). \tag{60}$$

Inserting (60) in (59) and using (55) we derive the following explicit representation of the solution of problem (56)

$$\begin{aligned} u(x,t) &= \frac{2P}{\pi}\sum_{n=1}^\infty \frac{1-(-1)^n}{n}\sin(n\pi x)\int_0^t\int_0^\infty \varphi(\tau,\sigma)\cos(n\pi\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \\ &= \frac{2P}{\pi}\sum_{n=1}^\infty \frac{1-(-1)^n}{n}\sin(n\pi x)\int_0^t S_n(\tau)\,\mathrm{d}\tau. \end{aligned}$$

This method can be easily generalized to a two-dimensional variant of problem (56) governing Poiseuille flow in a channel.

5. Numerical results

To provide an independent check of the explicit solutions found in Section 3 we also solve the corresponding problems numerically using an implicit finite difference method of first order accuracy in time and second order accuracy in space. For discretization of the fractional Riemann–Liouville derivatives the Grünwald–Letnikov approximation is used. The precise finite differences scheme can be found e.g. in [29]. Uniform spatial and temporal steps of order 10^{-3} are considered.

In all examples of this section $a = 1$ and $\mu = b/a$.

The solution of the Stokes' first problems in half-space is computed using representation (28). The results are plotted in Figs. 1 and 2. A comparison with the corresponding numerical solution obtained by finite differences is given in Fig. 1. It is clear that both methods give identical results. To illustrate the effect of the fractional parameter α on the solution behavior, in Fig. 2 solutions for different values of α are plotted. Let us note that for $\alpha = 1$ the solution is identical with that obtained in [28], see their Fig. 2.

For the computation of the solution of Stokes' first problem on a strip we use eigenfunction expansion (32)–(33), where the calculation of eigenmodes $S_n(t)$ for sufficiently large n is the most important and difficult part. Let us note that series expansion (33) can be used only for $t \le T_n$, where T_n decreases with increasing n. Therefore for large t we make use of the asymptotic expansion (37). In Fig. 3 plots of $S_n(t)$ are given for $\mu = 0.01$. For the computation of $S_n(t)$ we take 10^2 terms in any of the infinite sums: in k and for the computation of Mittag-Leffler functions (34). The behavior of $S_n(t)$ shown in Fig. 3 is in agreement with the conclusions in the end of Section 3.2. As expected, for this small value of μ we observe pronounced oscillations.

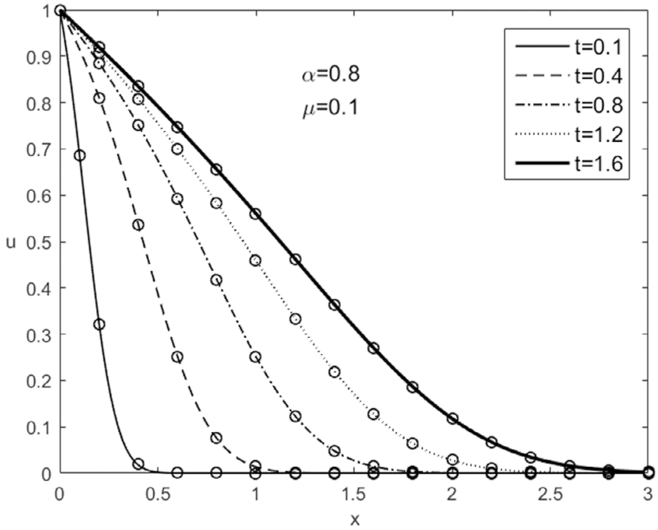

Fig. 1. The solution (28) of Stokes' first problem in half-space as a function of x for different values of t, compared to the numerical solution by finite difference method (symbols).

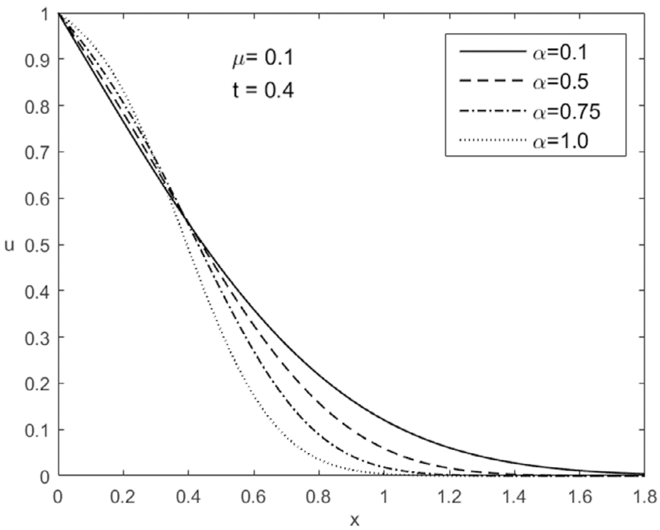

Fig. 2. The solution (28) of Stokes' first problem in half-space as a function of x for different values of α.

Figs. 4 and 5 present comparisons of the series solution (32) (with 15 terms) of Stokes' first problem on a strip and the corresponding numerical solution obtained by finite differences. Very good agreement between both solutions is observed. Let us also note that numerical results obtained from the series solution (32) in the case of classical Jeffreys' fluid ($\alpha = 1$) and $\mu = 0.5$ are in agreement with the results of [17], Fig. 1.

Fig. 6 shows another comparison concerning the solution of Stokes' first problem on a strip. The eigenmodes $S_n(t)$ in (32) are calculated in two different ways: by the integral formula (55) and by the series representation (33). Again, a very good agreement is observed. In this way, a numerical verification of the subordination formula is also provided.

6. Conclusions

It is established that the fractional Jeffreys' model is physically meaningful only if the orders of the two fractional derivatives in the constitutive equation coincide. This is due to the fact that only in this case the corresponding relaxation function is non-negative and non-increasing—properties related to the physical phenomenon of stress relaxation.

Based on the obtained constraints on the parameters the subordination principle is derived for a class of IBVPs governing the velocity distribution of unidirectional flows of fractional Jeffreys' fluids. The subordination identity splits the solution

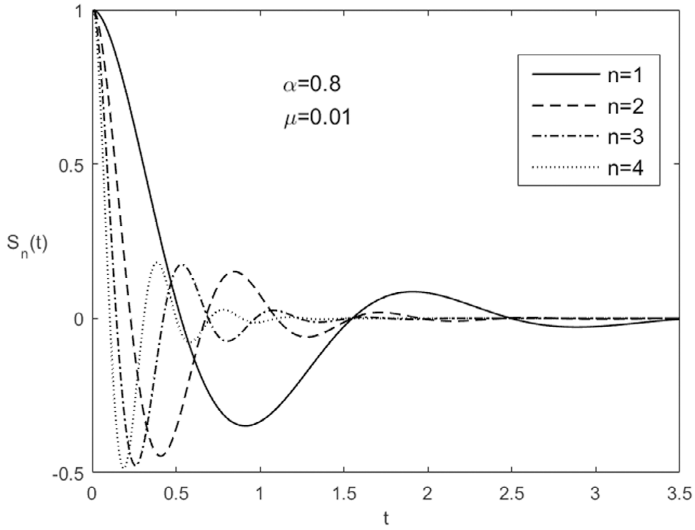

Fig. 3. Eigenmodes $S_n(t)$ computed using series expansion (33).

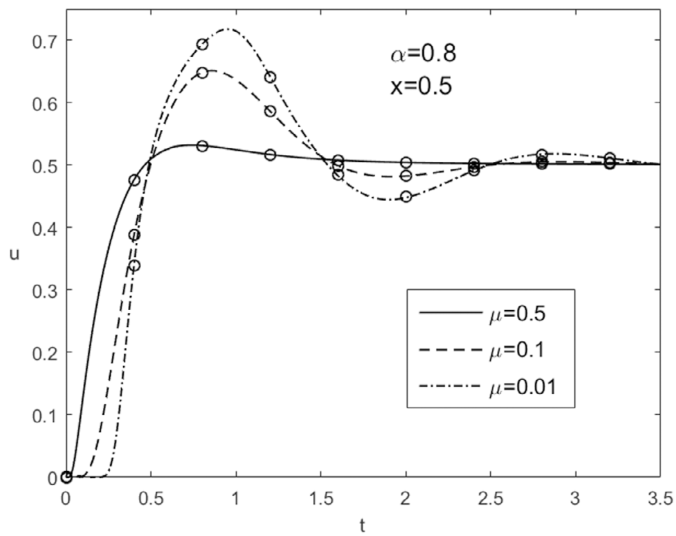

Fig. 4. The series solution (32) of Stokes' first problem on a strip as a function of t for different values of μ compared to the numerical solution by finite difference method (symbols).

into two parts. The first part (the p.d.f.) depends only on the constitutive model and the second part (the solution of a related wave equation) depends only on the flow geometry. Explicit representation of the p.d.f. is derived from the solution of the Stokes' first problem. The numerical verification of the obtained analytical results confirms their correctness.

The presented subordination approach can be generalized to Volterra integral equations with kernel $k(t)$, which Laplace transform $\widehat{k}(s)$ is well defined and positive for $s > 0$ and is such that $(\widehat{k}(s))^{-1/2}$ is a Bernstein function.

Acknowledgments

This work is partially supported by Bulgarian National Science Fund (Grant DFNI-I02/9); and performed in the frames of the bilateral research project between Bulgarian and Serbian academies of sciences (2014–2016) "Mathematical modeling via integral-transform methods, partial differential equations, special and generalized functions, numerical analysis".

Appendix

Here we list definitions and some properties of special type functions related to complete monotonicity.

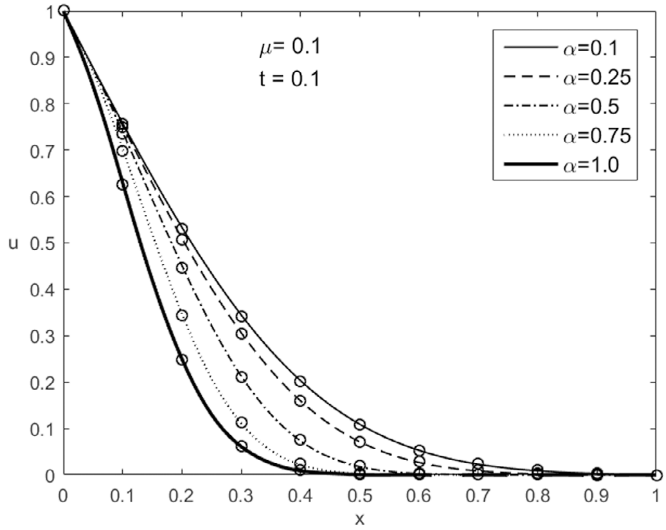

Fig. 5. The series solution (32) of Stokes' first problem on a strip as a function of x for different values of α compared to the numerical solution by finite difference method (symbols).

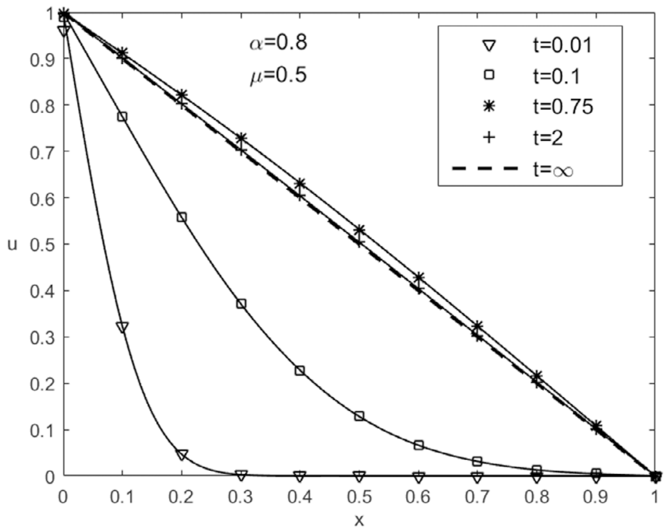

Fig. 6. The solution of Stokes' first problem on a strip as a function of x for different values of t obtained by subordination principle (lines) and using (32)–(33) (symbols).

A function $\varphi : (0, \infty) \to \mathbb{R}$ is said to be completely monotone function ($\varphi \in \mathcal{CMF}$) if it is of class C^∞ and

$$(-1)^n \varphi^{(n)}(\lambda) \geq 0, \quad \lambda > 0, \ n = 0, 1, 2, \ldots. \tag{61}$$

The Mittag-Leffler function $E_{\alpha,\beta}(-\lambda) \in \mathcal{CMF}$ for $0 < \alpha \leq 1, \alpha \leq \beta$ (see e.g. [31]).

The characterization of the class $\mathcal{CMF}$ is given by the Bernstein's theorem (see e.g. [32]) which states that a function is completely monotone if and only if it can be represented as the Laplace transform of a non-negative measure (non-negative function or generalized function).

The class of Stieltjes functions ($\mathcal{SF}$) consists of all functions defined on $(0, \infty)$ which can be written as a restriction of the Laplace transform of a completely monotone function to the real positive semi-axis. Obviously, $\mathcal{SF} \subset \mathcal{CMF}$.

A non-negative function φ on $(0, \infty)$ is said to be a Bernstein function ($\varphi \in \mathcal{BF}$) if $\varphi'(\lambda) \in \mathcal{CMF}$; $\varphi(\lambda)$ is said to be a complete Bernstein functions ($\mathcal{CBF}$) if and only if $\varphi(\lambda)/\lambda \in \mathcal{SF}$. We have the inclusion $\mathcal{CBF} \subset \mathcal{BF}$.

The following properties are satisfied:

(A) The class $\mathcal{CMF}$ is closed under point-wise addition and multiplication.

(B) If $\varphi \in \mathcal{BF}$ then $\varphi(\lambda)/\lambda \in \mathcal{CMF}$.

(C) If $\varphi \in \mathcal{CMF}$ and $\psi \in \mathcal{BF}$ then the composite function $\varphi(\psi) \in \mathcal{CMF}$.
(D) If $\varphi \in \mathcal{CBF}$ and $\psi \in \mathcal{SF}$ then the composite function $\varphi(\psi) \in \mathcal{SF}$.

For proofs and more details on these special classes of functions we refer to [33], see also [34,20].

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